Appendix G

Eligibility order reduction

The eligibility profile function is

$$\epsilon_i \quad = \quad C\, A^{i-1}\, B \qquad (i \geq 1) \tag{G.1}$$

where A has size $n_y \times n_y$, B has size $n_y \times n_x$, C has size $1 \times n_y$ and ϵ_i is a scalar.

It is sometimes necessary to reduce the order of this model by eliminating small eigenvalues. Small eigenvalues play an insignificant role in determining ϵ_i but they can cause great loss of numerical precision in the FOX algorithm. New, smaller matrices $\overline{A}$, $\overline{B}$ and $\overline{C}$ must be found such that $\overline{\epsilon}_i$ approximates ϵ_i:

$$\begin{aligned} \overline{\epsilon}_i \quad &= \quad \overline{C}\, \overline{A}^{\,i-1}\, \overline{B} \\ &\approx \quad \epsilon_i \end{aligned} \tag{G.2}$$

The matrix A can be decomposed into a diagonal matrix Λ of eigenvalues and a matrix S of eigenvectors as follows:

$$A \quad = \quad S\, \Lambda\, S^{-1} \tag{G.3}$$

which gives

$$\epsilon_i \quad = \quad [C\, S]\, \Lambda^{i-1}\, [S^{-1}\, B] \tag{G.4}$$

For example, if

$$A \quad = \quad \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.08 & 0.06 & 0.7 & 0.2 \\ 0.1 & -0.1 & 0 & 1 \end{bmatrix} \tag{G.5}$$

$$\tag{G.6}$$

then

$$\Lambda \quad = \quad \begin{bmatrix} 0.9847{+}0.0520i & 0 & 0 & 0 \\ 0 & 0.9847{-}0.0520i & 0 & 0 \\ 0 & 0 & 0.9718 & 0 \\ 0 & 0 & 0 & 0.7589 \end{bmatrix} \tag{G.7}$$

$$S = \begin{bmatrix} 0.4838{-}0.2314i & 0.4838{+}0.2314i & -0.7061 & -0.3790 \\ 0.6770{-}0.1641i & 0.6770{+}0.1641i & -0.6539 & -0.0556 \\ 0.0462{+}0.2870i & 0.0462{-}0.2870i & 0.1993 & 0.9139 \\ -0.0184{+}0.3772i & -0.0184{-}0.3772i & 0.1846 & 0.1341 \end{bmatrix} \tag{G.8}$$

The rows and columns of Λ and S corresponding to the smallest eigenvalues are removed, to obtain $\overline{\Lambda}$ and $\overline{S}$. In this example the fourth row and column are removed, corresponding to the smallest eigenvalue 0.7589 :

$$\overline{\Lambda} = \begin{bmatrix} 0.9847{+}0.0520i & 0 & 0 \\ 0 & 0.9847{-}0.0520i & 0 \\ 0 & 0 & 0.9718 \end{bmatrix} \tag{G.9}$$

$$\overline{S} = \begin{bmatrix} 0.4838{-}0.2314i & 0.4838{+}0.2314i & -0.7061 \\ 0.6770{-}0.1641i & 0.6770{+}0.1641i & -0.6539 \\ 0.0462{+}0.2870i & 0.0462{-}0.2870i & 0.1993 \end{bmatrix} \tag{G.10}$$

Now, for the remaining eigenvalues and eigenvectors to correctly combine to give $\overline{\epsilon}_i$, the reduced order model must be

$$\overline{\epsilon}_i = \overline{[C\,S]}\,\Lambda^{i-1}\,\overline{[S^{-1}\,B]} \tag{G.11}$$

where $\overline{[C\,S]}$ is the matrix CS with the chosen rows and columns removed, and similarly for $\overline{[S^{-1}\,B]}$. From equation G.4 the following must also be true:

$$\overline{\epsilon}_i = [\overline{C}\,\overline{S}]\,\overline{\Lambda}^{i-1}\,[\overline{S}^{-1}\,\overline{B}] \tag{G.12}$$

So by comparing equation G.12 and equation G.11 the reduced order model is

$$\overline{B} = \overline{S}\,\overline{[S^{-1}\,B]} \tag{G.13}$$

$$\overline{C} = \overline{[C\,S]}\,\overline{S}^{-1} \tag{G.14}$$

and

$$\overline{A} = \overline{S}\,\overline{\Lambda}\,\overline{S}^{-1} \tag{G.15}$$