Appendix E

Eligibility profile cookbook

The eligibility profile function is

$$\epsilon_i = C\,A^{i-1}\,B \qquad (i \geq 1) \tag{E.1}$$

A useful class of second order eligibility profiles is parameterized by the numbers k_a and k_b, and has the following matrices:

$$A = \begin{bmatrix} 1 & h \\ -hk_a & 1 - hk_b \end{bmatrix} \tag{E.2}$$

$$B = \begin{bmatrix} 0 \\ hk_a \end{bmatrix} \tag{E.3}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{E.4}$$

The matrices are derived from an Euler approximation (with step size h) of the following continuous differential equation

$$\ddot{\varepsilon} = k_a\,(s - \varepsilon) - k_b\,\dot{\varepsilon} \tag{E.5}$$

$$\varepsilon(0) = 0 \tag{E.6}$$

$$\dot{\varepsilon}(0) = h\,k_a \tag{E.7}$$

$$\text{where} \quad \epsilon_i = \varepsilon(ih) \tag{E.8}$$

Note that $\dot{\varepsilon}(0) = hk_a$ because the driving signal $s(t)$ is equal to the impulse $h\delta(t)$ (where $\delta(t)$ is the Dirac-delta function). The solutions for $\varepsilon(t)$ are

$$q = k_b^2 - 4\,k_a \tag{E.9}$$

$$\varepsilon(t) = \begin{cases} \dfrac{k_a}{\sqrt{q}} e^{(-k_b+\sqrt{q})t/2} - \dfrac{k_a}{\sqrt{q}}\, e^{(-k_b-\sqrt{q})t/2} & (\text{if} \quad q > 0) \\[2ex] \dfrac{2k_a}{\sqrt{-q}} e^{-k_b t/2} \sin\left(\dfrac{t\sqrt{-q}}{2}\right) & (\text{if} \quad q < 0) \\[2ex] k_a t\, e^{-k_b t/2} & (\text{if} \quad q = 0) \end{cases} \tag{E.10}$$

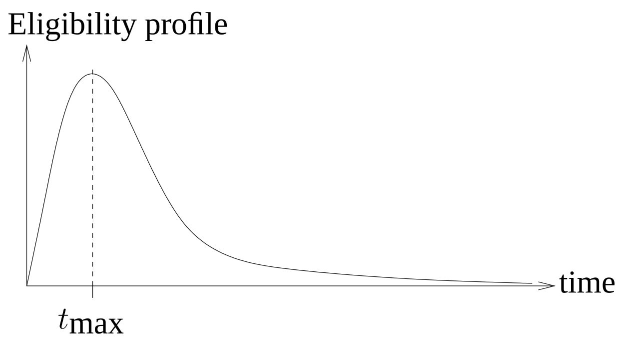

Figure E.1: *A second order critically damped eligibility profile.*

These solutions are respectively over-damped ($q > 0$), under-damped ($q < 0$) and critically damped ($q = 0$). The following table shows how to select k_a and k_b for these three cases.

Mode	Prototype profile	Parameters
Over-damped	$e^{-\tau_1 t} - e^{-\tau_2 t}$	$k_a = \tau_1 \tau_2$ and $k_b = \tau_1 + \tau_2$
Under-damped	$e^{-\tau t} \sin(\omega t)$	$k_a = \tau^2 + \omega^2$ and $k_b = 2\tau$
Critically damped	$t\, e^{-\tau t}$	$k_a = \tau^2$ and $k_b = 2\tau$

For the critically damped case another useful way to set the parameters is by $t_{\max}$, the time at which the eligibility profile reaches a maximum (see figure E.1). In this case

$$k_a = \frac{1}{t_{\max}^2} \tag{E.11}$$

$$k_b = \frac{2}{t_{\max}} \tag{E.12}$$